Optimal Savings (Ramsey Rule)

Divide $Q_t \nearrow C_t$, $\searrow K_{t+1} - K_t$ in each period.

$Q_t = F(K_t, N)$ production function, N constant.

Accounting $F(K_t, N) = C_t + \underbrace{[K_{t+1} - K_t]}_{\text{investment}}$

Find condition that yields a maximum of PV of welfare in

$$W = U(C_0) + \frac{1}{1+\rho} U(C_1) + \left(\frac{1}{1+\rho}\right)^2 U(C_2) + \left(\frac{1}{1+\rho}\right)^3 U(C_3) + \ldots + \infty$$

Replace each C_t from the accounting relation.

$$W = U\Big(F(K_0, N) - (K_1 - K_0)\Big) + \left(\frac{1}{1+\rho}\right) U\Big(F(K_1, N) - (K_2 - K_1)\Big) + \left(\frac{1}{1+\rho}\right)^2 U\Big(F(K_2, N) - (K_3 - K_2)\Big) + \ldots + \infty.$$

$$\frac{\partial W}{\partial K_1} = 0 \Rightarrow U_{c_0} \frac{dC_0}{dK_1} + \left(\frac{1}{1+\rho}\right) U_{c_1}\left[\frac{\partial C_1}{\partial F} \cdot \frac{\partial F_1}{\partial K_1} + \frac{\partial C_1}{\partial K_1}\right] = 0$$

$$U_{c_0}(-1) + \left(\frac{1}{1+\rho}\right) U_{c_1}\left[F_{K_1} + 1\right] = 0$$

This is the optimal savings Rule

$$-\frac{[U_{c_1} - U_{c_0}]}{U_{c_1}} + \frac{U_{c_0}}{U_{c_1}} \rho = F_{K_1}$$

in discrete time (periods 0 and 1 are t and $t+1$ in general)

$$-\frac{\frac{dU_c}{dt}}{U_c} + \rho = F_K$$

in continuous time

In continuous time START with

$$U_c(t) = \int_t^\infty U_c(s) F_k(s) e^{-\rho[s-t]} \, ds$$

$$\frac{dU_c(t)}{dt} = \rho U_c(t) - U_c(t) F_k(t)$$

Ramsey optimal savings Rule.

$U_c(t)$ is gain in C from 1 more unit of Q, valued in utils.

$\int_t^\infty U_c(s) F_k(s) e^{-\rho[s-t]} \, ds$ is gain from 1 more unit of Q in INVESTMENT valued in utils.